

SIMULATION, KRIGING, AND VISUALIZATION  
OF CIRCULAR-SPATIAL DATA

by

William J. Morphet

A dissertation submitted in partial fulfillment  
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematical Sciences

Approved:

---

Jürgen Symanzik  
Major Professor

---

Adele Cutler  
Committee Member

---

Donald H. Cooley  
Committee Member

---

John R. Stevens  
Committee Member

---

Daniel C. Coster  
Committee Member

---

Byron Burnham  
Dean of Graduate Studies

UTAH STATE UNIVERSITY  
Logan, Utah

2009

Copyright © William Morphet 2009

All Rights Reserved

## ABSTRACT

Visualization, Kriging, and Simulation  
of Circular-Spatial Data

by

William J. Morphet, Doctor of Philosophy  
Utah State University, 2009Major Professor: Dr. Jürgen Symanzik  
Department: Mathematics and Statistics

The circular dataimage is defined by displaying direction as the color at the same direction in a color wheel composed of a sequence of two-color gradients with color continuity between gradients. The resulting image of circular-spatial data is continuous with high resolution. Examples include ocean wind direction, Earth's main magnetic field, and rocket nozzle internal combustion flow. The cosineogram is defined as the mean cosine of the angle between random components of direction as a function of distance between observation locations. It expresses the spatial correlation of circular-spatial data. A circular kriging solution is developed based on a model fitted to the cosineogram. A method for simulating circular random fields is given based on a transformation of a Gaussian random field. It is adaptable to any continuous probability distribution. Circular random fields were implemented for selected circular probability distributions. An R software package was created with functions and documentation.

(391 pages)

## ACKNOWLEDGMENTS

Study of the dataimage of Minotte and West (1998), for the imaging of ordered multiple linear variables and observational units, motivated the problem of how to display directional-spatial data as an image, and what this method should be called. I am thankful for the helpful discussions with Dr. Mike Minnotte regarding color wheels. I would also like to thank the JCGS associate editor and referee for their helpful suggestions regarding circular dataimages. I acknowledge and express thanks to ATK Launch Systems, Inc., and NASA for authorizations to display model flow in the Space Shuttle solid rocket motor nozzle in Figure 2-17 (b) and to display nozzle vectoring data in Figure 2-18, and to Technical Artist Alan Eaton for Figure 2-19 (Property number A045477a, ATK Launch Systems, Inc., copyright © 2005) showing the Space Shuttle roll maneuver with left/right solid rocket motor labels added.

Most of the figures were generated in R, versions 2.8.0 (R Development Core Team 2008). R was originally created by Ihaka and Gentleman (1996) and is now a collaborative worldwide effort. The binary distributions of R and R contributor packages are freely downloadable from <http://www.r-project.org/>, and are supported on Windows (NT, 95 and later) and in some versions for other operating systems. Figure 1-4 was produced using functions of R package Fields (Furrer, Nychka, and Sain 2009), software for simulation of random fields. Figure 1-5, Appendix Figure N-1 (b), and Figure N-2 (a) were produced using functions of R package CircStats (Lund and Agostinelli 2007), software for circular statistics, and manually enhanced. The functions of CircStats were used extensively in the codes written for this dissertation. Figures N-2 (b) and (c) were produced using a demo version of Oriana 2, software for the analysis and display of circular data (Kovach Computing 2004), and manually enhanced. Oriana is available at <http://www.kovcomp.co.uk/oriana/oribroc.html>. Oriana is supported on Microsoft

Windows 98/Windows NT 4 or later, including Windows ME/2000/XP/2003/Vista.

Figures 2-15, 2-16, and N-4 were constructed using functions of R package RGL (Adler 2009), software for 3-D real time visualization. Other R packages used extensively include geoR (Ribeiro and Diggle 2001), and RandomFields (Schlather 2001).

I would like to acknowledge and thank:

ATK Launch Systems, Inc.:

- Dr. Suresh Kulkarni – Provided critical endorsement following rejection of application for the master's program of study.
- Management and supervision – Approved adjusted working hours to accommodate class schedules.

Utah State University professors:

- Dr. Donald Cooley – Head of Computer Science and defense chairman - Suggested programming to have some fun when needing a break.
- Dr. Dan Coster – Provided a listening ear and encouragement.
- Dr. Adele Cutler – Provided a critical evaluation leading to a productive topic of research and a thorough critique of my dissertation.
- Dr. Richard Cutler – Defended me and encouraged high GPA.
- Dr. Robert Heal – Provided a listening ear and encouragement.
- Dr. John Stevens – Suggested important improvements to the dissertation.
- Dr. Jürgen Symanzik, advisor – Provided guidance, persuaded me to remain in the doctoral program of study following withdrawals, and provided critical review.
- Dr. Russell Thompson - Was highly supportive as department head (without him, one withdrawal would have been terminal).

Others:

- Kenneth Johnson, NASA/ NESC Systems Engineering Office – For review of Chapter 1 of the dissertation.
- Terry, wife and mother of 5 children – Sacrificed many hours of companionship and performed many supportive efforts.

William J. Morphet

## CONTENTS

	Page
ABSTRACT .....	iii
ACKNOWLEDGEMENTS .....	iv
LIST OF TABLES .....	xii
LIST OF FIGURES .....	xiii
SYMBOLS, TERMINOLOGY, ACRONYMS .....	xxi
CHAPTERS	
1. INTRODUCTION	
1.1 Introduction to the Circular Random Field and Circular Random Variables .....	1
1.2 A Motivational Example .....	7
1.3 Problem Description .....	7
1.4 Literature Review .....	9
1.5 Dissertation Overview .....	15
2. CIRCULAR DATAIMAGE, A HIGH RESOLUTION CONTINUOUS IMAGE OF CIRCULAR-SPATIAL DATA	
2.1 Introduction .....	19
2.2 Overview of Vectorial-Spatial Displays .....	20
2.3 Cross Over .....	24
2.4 The Circular Dataimage and Color Wheel .....	25
2.5 Comparison of Methods .....	27
2.6 Calculation of a BGYR Color Wheel .....	32
2.7 Color Considerations and Variations .....	33
2.8 Other Examples .....	43
2.9 Chapter Summary and Future Work .....	50
3. COSINEOGRAM, A MEASURE OF CIRCULAR-SPATIAL CORRELATION	
3.1 Introduction .....	51
3.2 The Cosineogram .....	52
3.3 Derivation of the Sill .....	54
3.4 Expectation of the Cosines .....	59
3.5 Verification of the Sill by Simulation .....	63
3.6 Cosine Models .....	67
3.7 Cosineogram of Ocean Wind in a South Polar Region .....	73
3.8 Chapter Summary and Future Work .....	75

## CONTENTS

	Page
4. CIRCULAR KRIGING	
4.1 Introduction.....	78
4.2 Solution .....	79
4.3 Verification of Optimality .....	88
4.4 Computationally Efficient Formula .....	90
4.5 Kriging Behavior Around a Sampled Location.....	91
4.6 Circular Kriging Variance, $\sigma_{CK}^2$ .....	94
4.7 How Distance and the Cosine Model Affect $\hat{\sigma}_{CK}^2$ .....	98
4.8 Chapter Summary and Future Work .....	101
5. SIMULATION OF CIRCULAR RANDOM FIELDS	
5.1 Introduction.....	103
5.2 Background .....	104
5.3 New Method of Generating a CRF.....	108
5.4 Mathematical Properties of the CRF .....	112
5.5 Qualitative Evaluations of Method of Simulating a CRF .....	124
5.6 Extension of the Method .....	128
5.7 Chapter Summary and Future Work .....	129
6. COMPREHENSIVE EXAMPLE	
6.1 Outline of Circular-Spatial Processes .....	131
6.2 Simulation of a CRF .....	131
6.3 Estimation of the Spatial Trend.....	133
6.4 Computation of the Residuals.....	134
6.5 Plotting and Modeling the Cosineogram .....	135
6.6 Kriging the Residuals.....	136
6.7 Interpolation of the Trend Estimate .....	137
6.8 Computing The Circular-Spatial Estimate .....	138
6.9 Imaging the Circular-Spatial Estimate.....	139
6.10 Computing the Circular Kriging Variance .....	139
7. SUMMARY.....	141
CITATIONS .....	147
APPENDICES .....	151
A Notation .....	152



## CONTENTS

	Page
B    Linear Algebra	
B.1   Identities for Vectors.....	153
B.2   Some Properties of the Positive Definite Matrix <b>K</b> .....	153
B.3   Theorem: The P. D. Matrix Has an Inverse.....	154
B.4   Theorem: The Inverse of P. D. Matrix Is Symmetric.....	154
B.5   Theorem: The Inverse of P. D. Matrix Is P. D. ....	155
B.6   Some Properties of the Negative Definite Matrix .....	155
B.7   Derivatives Required for Kriging .....	156
B.8   The Requirements for Maximization .....	158
B.9   Expectation.....	159
C    Qualitative Evaluations of Other CRFs with Standardization .....	160
D    Qualitative Evaluations of CRFs Near Parameter Extremes .....	165
E    Derivations of the CDF Formulae for Support $[0, 2\pi)$	
E.1   Cardioid.....	184
E.2   Triangular .....	184
E.3   Uniform.....	186
E.4   Von Mises .....	186
E.5   Wrapped Cauchy.....	187
F    Verification by Evaluation of the CDF Formulae with Support $[0, 2\pi)$	
F.1   Cardioid.....	188
F.2   Triangular .....	188
F.3   Uniform.....	189
F.4   Von Mises .....	189
F.5   Wrapped Cauchy.....	190
G    Modification of the PDF and CDF Formulae for Rotated Support $[-\pi, +\pi)$	
G.1   Cardioid.....	192
G.2   Triangular .....	193
G.3   Uniform.....	194
G.4   Von Mises .....	194
G.5   Wrapped Cauchy.....	195
H    Wrapped Cauchy CDF	
H.1   Additional Forms of the CDF .....	196
H.2   Alternate Forms with Support $[0, 2\pi)$ .....	198

## CONTENTS

	Page
H.3 Evaluation of Alternate Forms .....	199
H.4 Selected Form for Rotated Support $[-\pi, +\pi)$ .....	201
I Triangular Inverse CDF .....	202
J R Package Documentation	
J.1 Introduction and Installation .....	204
J.2 SimulateCRF .....	206
J.3 CircResidual .....	211
J.4 CosinePlots .....	214
J.5 KrigCRF .....	222
J.6 InterpDirection .....	228
J.7 TestPattern .....	233
J.8 OceanWind .....	234
J.9 WorldMask .....	234
J.10 CircDataimage .....	235
J.11 PlotVectors .....	244
K R Functions	
K.1 TestPattern .....	248
K.2 CircDataimage .....	249
K.3 SimulateSill .....	271
K.4 CorrelationTransfer .....	273
K.5 SimulateCRF .....	274
K.6 AssessCRF .....	278
K.7 PlotVectors .....	282
K.8 CircResidual .....	284
K.9 CosinePlots .....	285
K.10 KrigCRF .....	288
K.11 InterpDirection .....	291
K.12 CircMedianPolish .....	296
K.13 AssessStandardization .....	298
K.14 MakeCosineData .....	302
K.15 FitCosineData .....	305
K.16 FitOceanWind .....	306
K.17 3DPolarMainMagnetic .....	307
K.18 Circular Kriging Variance .....	310
L R Command Line Input	
L.1 Figures 3-5 to 3-9 .....	311
L.2 Figure 3-13 .....	312
L.3 Figure 5-1, Simulated CRF .....	314

## CONTENTS

	Page
L.4 Figure 5-3, Image of GRF .....	314
L.5 Figure 5-4, Variogram and Inverted Cosineogram Similar .....	314
L.6 Figure 5-5, Standardization .....	315
L.7 Figure 5-6, Variability vs. $\rho$ .....	315
L.8 Figure 5-8 and the Figures in Appendices C and D. ....	315
L.9 Figures 6-2 to 6-11 .....	317
L.10 Make Cosine Datasets .....	321
L.11 Figure M-1, Fitted Covariogram an Unbiased Estimator .....	326
L.12 Plot Figures M-2, M-3, and M-4, Families of Curves .....	327
L.13 Plot Figures M-6 to M-10 .....	333
L.14 Plot Figures 4-3 and 4-4 .....	336
 M Cosine Curves of Simulated Circular Random Fields (CRF)	
M.1 Review .....	338
M.2 Generation of Cosine vs. Distance Curves .....	338
M.3 Families of Cosine Curves .....	341
M.4 Characterization of the Cosine Curves .....	345
M.5 Expressions for the Cosine Models of Table M-2.....	348
M.6 Generalization of the Generation and Characterization of the Cosine Curves .....	353
 N Additional Graphics for Circular Data	
N.1 Summary Plots for Circular Data .....	354
N.2 Histograms for Circular Data .....	355
N.3 Nonparametric Density Plots for Circular Data.....	357
N.4 New Cylindrical Plot of the Circular Probability Density.....	358
 O Permissions .....	360
 CURRICULUM VITAE .....	368

## LIST OF TABLES

Table	Page
2-1 BGYR Color Wheel Formulae for RGB Space .....	32
3-1 Circular Probability Distributions, $\mu = 0$ , $0 \leq \theta < 2\pi$ Radians .....	56
3-2 The Sill of Selected Distributions.....	64
5-1 Circular Probability Distributions, $\mu = 0$ , $-\pi \leq \theta < \pi$ Radians .....	111
5-2 CDFs and Inverse CDFs for Circular Distributions, $\mu = 0$ , $-\pi \leq \theta < \pi$ Radians .....	112
D-1 Spatial Property Scores of Figures D-1 to D-16 .....	182
J-1 Output of CosinePlots .....	220
J-2 Output of KrigCRF .....	224
J-3 Output of PlotVectors.....	245
M-1 Mean Resultant Vector Length $\rho$ of Circular Distributions for Figures M-2, M-3, and M-4 .....	339
M-2 Cosine Models Approximating CRF Cosine Curves .....	347

## LIST OF FIGURES

Figure	Page
1-1 Circular PDF of the Triangular Circular Probability Distribution .....	2
1-2 The Arithmetic Mean of $180^\circ$ Does Not Point in the Central Direction of $0^\circ$ .....	6
1-3 The Effect of the Population Resultant Vector Mean Length $\rho$ on the Sample Mean Resultant Vector (Black) of a Sample (Tan) from the von Mises Circular Distribution.....	6
1-4 Circular and Vector Spatial Data and Their Means for the Direction the Ocean Wind Blows Toward .....	8
1-5 Rose Plot of the Circular Data Derived from the Data of Figure 1-3 .....	11
1-6 Kriging, Estimation of Spatial Data Based on Spatial Correlation .....	14
1-7 Flow Chart of Methods for Circular-Spatial Data .....	17
2-1 Circular Dataimage of the Direction Wind Is Blowing Toward, Coded with Yellow-Red-Green-Blue (YRGB) Color Wheel (Right) .....	19
2-2 Some Existing Methods for Display of Circular and Vectorial-Spatial Data Using the Smoothed Ocean Wind Data .....	23
2-3 Evolution of the YRGB Color Wheel.....	25
2-4-1 Plots of Average Ocean Wind Direction .....	28
2-4-2 Plots of Smoothed Ocean Wind Direction .....	29
2-5 Comparison of Arrow and Circular Dataimage Plots of Ocean Wind Average Direction .....	31
2-6 Normal and Simulated Deuteranopic Views of Images.....	35
2-7 Variety of Continuous and Discrete Color Wheels.....	36
2-8 Effects of GYRB Color Wheel and Smoothness of Data.....	38
2-9 Effects of Color Wheel Rotation, Color Wheel Labeled with Rotation.....	38
2-10 Focus Plots of Smoothed Average Direction with Focal Directions $0^\circ$ (Top) and $180^\circ$ (Bottom) .....	40

## LIST OF FIGURES

Figure		Page
2-11	Axial Focus Plots of Smoothed Average Direction with Axial Focal Directions 0 ° (Top) and 90 ° (Bottom) .....	41
2-12	Strength Binned by Quartiles and Coded as Value (V) in HSV Scheme.....	42
2-13	Circular Dataimage of Wind with Direction Coded Using HSV Color Wheel and Magnitude (m/s) Plotted as Contour Curves.....	43
2-14	Circular Dataimage of Earth Main Magnetic H Field Direction .....	44
2-15	3D Polar Plot of Earth Main Magnetic H Field Model with Direction as a Color and Magnitude as Radius for 1/1/1900, 1/1/1950, and 1/1/2000 .....	45
2-16	Asymmetry of Earth Main Magnetic H Field Model 1/1/2000 Demonstrated by 45° Rotations about the Horizontal Axis Through 0°-180° Longitude at the Equator .....	45
2-17	Space Shuttle Booster, Nozzle, and Nozzle Internal Combustion Flow .....	46
2-18	Time Series of the Space Shuttle Booster Nozzle Direction Angle .....	48
2-19	Illustration of the Space Shuttle Roll Maneuver vs. Time from Ignition .....	49
3-1	Distance Between Locations vs. Angular Distance Between Observations.....	52
3-2	Features of the Cosine Model .....	53
3-3	Acute and Obtuse Cases of Random Directions .....	58
3-4	Mean Cosine of the Angle Between Independent Cardioid CRV, $\rho^2 = 0.062$ , Is Consistent with the Theoretical Sill .....	65
3-5	Mean Cosine of the Angle Between Independent Triangular CRV, $\rho^2 = 0.041$ , Is Consistent with the Theoretical Sill .....	65
3-6	Mean Cosine of the Angle Between Independent Uniform CRV, $\rho^2 = 0$ , Is Consistent with the Theoretical Sill .....	66
3-7	Mean Cosine of the Angle Between Independent Von Mises CRV, $\rho^2 = 0.798$ , Is Consistent with the Theoretical Sill .....	66

## LIST OF FIGURES

Figure	Page
3-8 Mean Cosine of the Angle Between Independent Wrapped Cauchy CRV, $\rho^2 = 0.135$ , Is Consistent with the Theoretical Sill .....	67
3-9 The Exponential Cosine Model .....	69
3-10 The Gaussian Cosine Model .....	70
3-11 The Spherical Cosine Model .....	70
3-12 Circular Dataimage of Model of Ocean Wind Direction for South Polar Region.....	74
3-13 Cosineocloud, Cosineogram, and Exponential Model of South Polar Ocean Wind.....	75
4-1 Circular Kriging, the Interpolation of Circular-Spatial Data Based on Spatial Correlation .....	78
4-2 Directions Represented by the Unobserved $\mathbf{u}_0$ , Estimate $\hat{\mathbf{u}}_0$ , and Error $\mathbf{e}_0 = \hat{\mathbf{u}}_0 - \mathbf{u}_0$ Vectors.....	81
4-3 Effect of Cosine Model on the Kriging Estimate Around the Measurement Location.....	92
4-4 Effect of Range, Mean Resultant Length $\rho$ , and nugget $n_g$ on the Circular Kriging Variance $\hat{\sigma}_{CK}^2$ .....	100
5-1 Simulated Sample of a von Mises CRF, $\rho = 0.8$ , Range $r = 10$ .....	108
5-2 Mapping a GRV to a CRV via the CDFs $F_Z$ and $G_\Theta$ .....	109
5-3 Simulated GRF with Spherical Covariance Model and Range $r = 10$ Corresponding to Figure 5-1 .....	109
5-4 Similar Shapes of Variograms and Inverted Cosineogram Reflect Transformations of the Spatial Correlation of the GRF .....	116
5-5 Standardization of the GRV Increases Fit of the GRV and the CRV.....	121
5-6 Variability of Fit of the Simulated Triangular CRV Increases as $\rho$ Decreases .....	123

## LIST OF FIGURES

Figure	Page
5-7 Standardization of the GRV Biases GRF the Covariance.....	124
5-8 Evaluation of a von Mises CRF, $\rho = 0.8$ , Overfit, Range $r = 10$ .....	126
6-1 Comprehensive Example - The Trend Model, or the Underlying First Order Component of Variation .....	132
6-2 Comprehensive Example - Simulated Sample of a von Mises CRF, $\mu = 0, \rho = \sqrt{0.5}$ with Underlying Trend.....	132
6-3 Comprehensive Example - Comparison of the Trend Estimate (Tan) with the True Trend (Blue) .....	133
6-4 Comprehensive Example - Enlarged View of the Data (Black), Trend Estimate (Tan), and Residual Rotation (Dashed Red) Corresponding to the Green Highlighted Area in Figures 5-9 to 5-11 .....	134
6-5 Comprehensive Example - Points of the Cosineogram, and the Exponential, Gaussian, and Spherical Cosine Models of Circular-Spatial Correlation. ....	135
6-6 Comprehensive Example - Enlarged View of the Kriging (Light Grey) and the Residual Rotations (Red) Corresponding to the Green Highlighted Area in Figures 5-9 to 5-11 .....	136
6-7 Comprehensive Example - Enlarged View of the Interpolation (Purple) of the Trend Estimate (Tan) Corresponding to the Green Highlighted Area in Figures 5-9 to 5-11 .....	137
6-8 Comprehensive Example - Enlarged View of the Circular Spatial Data Estimate (Gold) and the Sample (Black) Corresponding to the Green Highlighted Area in Figures 5-9 to 5-11. ....	138
6-9 Comprehensive Example – Circular Dataimage (Left) of the Circular Spatial Data Estimate with HSV Color Wheel (Right) of Direction. ....	139
6-10 Comprehensive Example - Circular Kriging Variance with Measurements on a Regular Grid .....	140
C-1 Evaluation of a Cardioid CRF, $\rho = 0.25$ , Overfit, Range $r = 10$ .....	161



## LIST OF FIGURES

Figure	Page
C-2 Evaluation of a Triangular CRF, $\rho = 0.203$ , Overfit, Range $r = 10$ .....	162
C-3 Evaluation of a Uniform CRF, Overfit, Range $r = 10$ .....	163
C-4 Evaluation of a Wrapped Cauchy CRF, $\rho = 0.5$ , Overfit, Range $r = 10$ .....	164
D-1 Evaluation of a Cardioid CRF, $\rho = 0.05$ , Overfit, Range $r = 10$ .....	166
D-2 Evaluation of a Cardioid CRF, $\rho = 0.05$ , Range $r = 10$ .....	167
D-3 Evaluation of a Cardioid CRF, $\rho = 0.475$ , Overfit, Range $r = 10$ .....	168
D-4 Evaluation of a Cardioid CRF, $\rho = 0.475$ , Range $r = 10$ .....	169
D-5 Evaluation of a Triangular CRF, $\rho = 0.05$ , Overfit, Range $r = 10$ .....	170
D-6 Evaluation of a Triangular CRF, $\rho = 0.05$ , Range $r = 10$ .....	171
D-7 Evaluation of a Triangular CRF, $\rho = 0.385$ , Overfit, Range $r = 10$ .....	172
D-8 Evaluation of a Triangular CRF, $\rho = 0.385$ , Range $r = 10$ .....	173
D-9 Evaluation of a von Mises CRF, $\rho = 0.05$ , Overfit, Range $r = 10$ .....	174
D-10 Evaluation of a von Mises CRF, $\rho = 0.05$ , Range $r = 10$ .....	175
D-11 Evaluation of a von Mises CRF, $\rho = 0.95$ , Overfit, Range $r = 10$ .....	176
D-12 Evaluation of a von Mises CRF, $\rho = 0.95$ , Range $r = 10$ .....	177
D-13 Evaluation of a Wrapped Cauchy CRF, $\rho = 0.05$ , Overfit, Range $r = 10$ .....	178
D-14 Evaluation of a Wrapped Cauchy CRF, $\rho = 0.05$ , Range $r = 10$ .....	179
D-15 Evaluation of a Wrapped Cauchy CRF, $\rho = 0.95$ , Overfit, Range $r = 10$ .....	180
D-16 Evaluation of a Wrapped Cauchy CRF, $\rho = 0.95$ , Range $r = 10$ .....	181

## LIST OF FIGURES

Figure	Page
G-1 Visual Verification of Cardioid CDF, $\rho = 0.30$ , Support $[-\pi, +\pi)$ Radians .....	192
G-2 Visual Verification of Triangular CDF, $\rho = 0.30$ , Support $[-\pi, +\pi)$ Radians .....	193
G-3 Visual Verification of von Mises CDF, $\rho = 0.30$ , Support $[-\pi, +\pi)$ Radians .....	195
H-1 Incorrect Wrapped Cauchy CDF, $\rho = 0.75$ , Support $[0, 2\pi)$ Radians .....	196
H-2 Dataplot WCACDF of Wrapped Cauchy CDF, $\rho = 0.75$ , Support $[0, 2\pi)$ Radians.....	197
H-3 Three Forms of the Wrapped Cauchy CDF, $\rho = 0.75$ , Support $[-\pi, +\pi)$ Radians.....	199
H-4 Iterated Wrapped Cauchy CDF, $\rho = 0.95$ , Support $[-\pi, \pi)$ Radians, 15 Iterations .....	200
H-5 Visual Verification of Wrapped Cauchy CDF, $\rho = 0.75$ , Support $[-\pi, \pi)$ Radians .....	201
I-1 Visual Verification of Triangular Inverse CDF, $\rho = 0.95 * 4 / \pi^2$ , Support $[-\pi, +\pi)$ Radians.....	203
J-1 Mapping a GRF to a CRF via CDFs .....	207
J-2 Shapes of Variograms and Inverted Cosineogram Show Spatial Correlation Transformed from the GRF with Spherical Covariance and Range $r = 10$ .....	208
J-3 Plots of True Model, Simulated CRF, Data, Fitted Model, and Residuals.....	213
J-4 Distance Between Locations (Red) vs. Angular Distance (Grey) Between Observations.....	214
J-5 Features of the Cosineogram Model .....	215
J-6 Cosineocloud .....	215

## LIST OF FIGURES

Figure		Page
J-7	Empirical Cosineogram .....	216
J-8	Cosine Models for Circular-Spatial Data, Range $r = 8$ .....	218
J-9	Fitted Cosine Models .....	222
J-10	Residual Rotations (Black) Overplotted on the Circular Kriging (Tan) .....	224
J-11	Smoothing via the Nugget Not Effective at Data Locations .....	225
J-12	Smoothing the Kriging Components Is Effective at All Locations.....	226
J-13	Variability of the Circular Kriging Estimate with Locations on a Regular Grid .....	227
J-14	Variability of the Circular Kriging Estimate with Random Locations.....	228
J-15	Six Cases of Interpolation Location Indicated by Labeled Red Dots.....	229
J-16	Effect of Interpolation on Smoothed Average Wind Direction with BGYR Color Wheel.....	229
J-17	Fitted Model (Black) Overplotted on the Fitted Model Interpolation (Tan).....	231
J-18	Original Data (Black) Overplotted on the Estimates (Tan) .....	232
J-19	Enlargement of Figure J-16.....	233
J-20	Image Plot of WorldMask .....	235
J-21	Comparison of Arrow and Circular Dataimage Plots of Ocean Wind Average Direction .....	236
J-22	Evolution of the YRGB Color Wheel.....	238
J-23	Initial Display of the GUI, the Circular Dataimage Window (R Graphics Device 2), and the Color Wheel Window .....	239
J-24	Display with Circular Dataimage of Average Direction after Inputs Entered...	240
J-25	GYRB Color Wheel Rotated 90°, Data Smoothed with Bandwidth 2.5, and Display Coordinates Changed (Zoomed) .....	242
J-26	HSV Color Wheel Rotated 90°, Data Smoothed with Bandwidth 2.5, Color Scale Gap 0.20, and Arrows on .....	243

## LIST OF FIGURES

Figure	Page
J-27 Mask Restores Land Mass Shapes in Smoothed Data .....	244
J-28 Unit Vector Plot of Ocean Wind Data .....	246
J-29 Vector Plot of Ocean Wind Data .....	247
J-30 Triangle Icon Plot of Ocean Wind Data .....	247
M-1 Fitted Covariogram an Unbiased Estimator of Spherical Covariance .....	340
M-2 Family of Cosine vs. Distance Curves from the GRF with Exponential Covariance .....	342
M-3 Family of Cosine vs. Distance Curves from the GRF with Gaussian Covariance .....	343
M-4 Family of Cosine vs. Distance Curves from the GRF with Spherical Covariance .....	344
M-5 Whittlematern Cosine Model ( $\alpha=.493$ ) Approximates the Cosine Curve of the von Mises CRF, $\rho = 0.95$ , Transformed from an Exponential GRF, Range=5 .....	346
M-6 Whittlematern Cosine Models for $\rho = 0$ .....	348
M-7 Cauchytm Cosine Models for $\rho = 0$ .....	349
M-8 Generalized Cauchy Cosine Models for $\rho = 0$ .....	350
M-9 Hyperbolic Cosine Models for $\rho = 0$ .....	351
M-10 Stable Cosine Models for $\rho = 0$ .....	352
N-1 Summary Plots of the Ocean Wind Data.....	354
N-2 Circular Histograms of the Ocean Wind Data.....	356
N-3 Kernel Density Plots of the Ocean Wind Data .....	358
N-4 New Cylindrical Plot of PDFs of von Mises Probability Densities.....	359

## LIST OF SYMBOLS, TERMINOLOGY, ACRONYMS

## Symbols

- $\sigma_{CK}^2$  : Circular kriging variance  
 $\Theta$  : Circular random variable (CRV)  
 $\theta$  : Observation (realization) or simulation of a CRV  
 $\mathbf{C}$  : Matrix of cosines of angles between observations of direction  
 $\mathbf{c}$  : Vector of cosines of angles between observations and unobserved direction to be estimated  
 $\varsigma(d)$  : Model of the mean cosine of the angle between random components of direction as a function of distance between observation locations  
 $\hat{\varsigma}(d)$  : Cosineogram estimate of  $\varsigma(d)$   
 $\kappa$  : Concentration parameter of the von Mises distribution  
 $n_g$  : Nugget  
 $\kappa$  : Population concentration about the mean direction for von Mises CRV  
 $\mu$  : Population mean resultant vector direction  
 $\rho$  : Population resultant vector mean length and concentration about the mean direction  
 $\overline{R}_n$  : Sample resultant mean vector length  
 $\mathbf{x}$  : Vector of spatial coordinates

## New Terminology

Circular Dataimage  
 Cosineocloud  
 Cosineogram  
 Cosine Model  
 Circular Random Field

## Terminology from Linear Kriging

Covariogram  
 Covariance Model  
 Nugget  
 Range  
 Sill  
 Variogram

## LIST OF SYMBOLS, TERMINOLOGY, ACRONYMS

## Acronyms

CDF : Cumulative Distribution Function  
CRF : Circular Random Field  
CRV : Circular Random Variable  
CCW: Counterclockwise  
GRV : Gaussian Random Variable  
GRF : Gaussian Random Field  
GUI : Graphical User Interface  
GYRB: Green Yellow Red Blue  
HSV : Hue Saturation Value  
KBWR: Black Blue White Red  
MAD : Mean Absolute difference  
PDF : Probability Density Function  
RF : Random Field  
RGB : Red Green Blue  
RV : Random Variable  
YRGB: Yellow Red Green Blue